MA 222 - Analysis II: Measure and Integration (JAN-APR, 2016)

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Problem set 4

1. Construct an example to show that the a.e convergence need not imply convergence in measure in an infinite measure space.

2. Give an example to show that the convergence in measure need not imply a.e. convergence.

3. Let $f_n \to f$ in measure in E and g be a measurable function which is finite a.e. Show that $f_n \to g$ in measure if and only if $f = g$ a.e.

4. Let E be of finite measure, $f_n \to f$ in measure and g be measurable and finite a.e. Prove that $f_n g \to fg$ in measure and further, if $g_n \to g$ in measure, then $f_n g_n \to fg$ in measure.

5. Let $\mu(E) < \infty$ and define $\rho(f,g) = \displaystyle\int_E \frac{|f-g|}{1+|f-g|}$. Show that $f_n \to f$ in measure if and only if $\rho(f_n, f) \to 0$.

6. Let δ_{x_0} be the Dirac measure at x_0 defined on $([0,1],\ \mathcal{P}([0,1]))$, $x_0 \in [0,1]$. For $0 \le t \le 1$, define $\mu_n^t =: \displaystyle\sum_{k=0}^{n} {}^nC_k\ t^k (1-t)^{n-k} \delta_{k/n}$

   (a) Prove the following that μ_n^t is a measure satisfying $\mu_n^t([0,1]) = 1$, $\displaystyle\int_0^1 x\ d\mu_n^t = t$ and $\displaystyle\int_0^1 (x-t)^2\ d\mu_n^t = \frac{t(1-t)}{n}$.

   (b) For any $\epsilon > 0$, prove $\mu_n^t(A_\epsilon) = \dfrac{t(1-t)}{n\epsilon^2}$, where $A_\epsilon = \{x \in [0,1] : |x-t| \ge \epsilon\}$.

   (c) Let $f \in [0,1]$, Show that $\displaystyle\int_0^1 f\ d\mu_n^t \to f(t)$ uniformly in t.

   (d) Deduce Weierstrass approximation theorem: any continuous function in $[0,1]$ can be approximated uniformly by polynomials.

   (e) Let $f \in C[0,1]$ and $\displaystyle\int_0^1 x^n f(x)\ dx = 0,\ n = 0,1,2,\cdots$ then $f \equiv 0$.

7. Prove the following: Let $f, g : [a, b] \to \mathbb{R}$ measurable and $c \in \mathbb{R}$:

   (a) $T_a^b f = T_a^c f + T_c^b f,\ a \le c \le b$

   (b) $T_a^b(f+g) \le T_a^b f + T_a^b g$

   (c) $T_a^b(cf) = |c|\ T_a^b f$

8. If $f \in \mathbf{BV}[a, b]$, show that f' exists, integrable and $\int_a^b |f'| \le T_a^b f$.

9. Prove $x^\alpha \cos\left(\frac{1}{x^\beta}\right)$ defined on $(0,\ 1)$ is of $\mathbf{BV}(0, 1)$ if $\alpha > \beta > 0$. Also prove that $x^\alpha \cos\left(\frac{1}{x^\alpha}\right)$ is not of $\mathbf{BV}[0, 1]$.

10. Let $f \in C^1[a, b]$ such that $|f'(x)| \le M,\ \forall\ x$, then f is absolutely continuous.

11. Let f be continuous on [0,1] and absolutely continuous in $[\epsilon, 1],\ \forall\ \epsilon > 0$. Is f absolutely continuous on [0,1]? What if f is of $\mathbf{BV}[0, 1]$?

12. Construct an absolutely continuous, strictly increasing function g on [0,1] such that $g' = 0$ on a set of positive measure.